Slope Fields in AP Calculus
GA²PMT mini-conference -- [redacted]

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What is a slope field?

In the simplest terms, a slope field is the graphical representation of all possible solutions to a differential equation. In this way, a slope field is very much like a graphical antiderivative. It is very important to remember that ***functions*** (relations) are the solutions to both differential equations and antiderivatives. Since a slope field is only a graphical representation of a differential equation, it should **never** be considered a **proof** of anything specific about the function in question or its derivative (i.e. point, slope, concavity, or equation(s)).

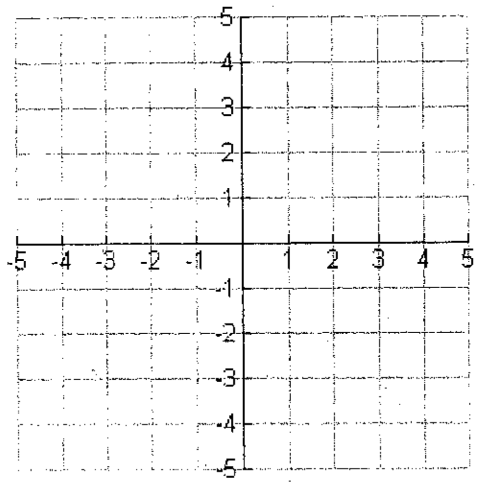
Why are they troublesome?

A difference that throws many students is that differential equations don't often look like antiderivative problems. Rather than asking a student to solve $\int 2x\,dx$, they are asked to find the function, $f(x)$, satisfying $\frac{dy}{dx} = 2x$. Seen another way, however, this is the **core** of the antiderivative concept - playing Jeopardy! I'm giving you the derivative and challenging you to determine the form of the original function. A differential equation gives you a derivative (same as an antiderivative), so all you have to do is identify the original.... Simple enough?

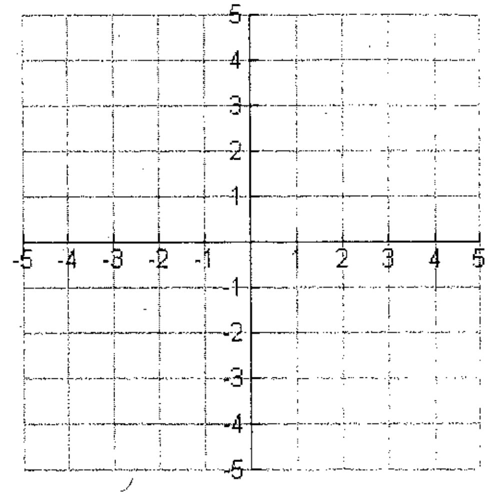
Since the only differential equation solution technique required by the AP is separable variables, the task is even easier; either you recognize that $2x$ is the derivative of anything in the form of $f(x) = x^2 + C$ or you separate the variables and end up with the equivalent antiderivative:

$$\frac{dy}{dx} = 2x \;\Rightarrow\; 1\,dy = 2x\,dx \;\Rightarrow\; \int 1\,dy = \int 2x\,dx \;\Rightarrow\; y = x^2 + C$$

Sketch multiple solutions to $f(x) = x^2 + C$



Sketch a slope field for $\frac{dy}{dx} = 2x$



Creating your own

Slope fields become more troublesome when their algebraic form is the result of implicit differentiation. One easy example is: $x^2+y^2=5 \Rightarrow 2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-x}{y}$. Assuming you were only given the final result $\left(\frac{dy}{dx}=\frac{-x}{y}\right)$ and did not recall or want to do separation of variables, what could you do? **Graph the slopes to get an <u>idea</u> about the look of the solutions.**

In general, I encourage my students to follow four basic guidelines: 1) Where is the slope zero, 2) Where is the slope undefined, 3) Where is the slope ± 1, and 4) Where is the slope positive and/or negative?

Slope field criteria

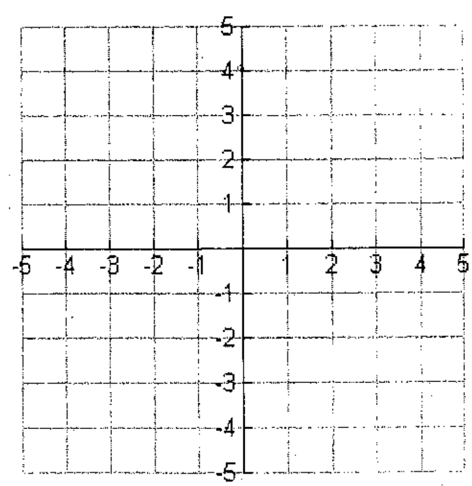
1)

2)

3)

4)

Sketch a slope field for $\frac{dy}{dx} = \frac{-x}{y}$



Notice that the function solutions to $\frac{dy}{dx}=\frac{-x}{y}$ are no longer simple vertical translations of a parent function; that only happens when $\frac{dy}{dx}$ can be expressed as a function of x only. Looking at your last picture, one might suspect circles to be the solutions: $x^2+y^2=r^2$. Draw two different solutions on the slope field - start one from **(1,0)** and another from **(-2,2)**. Thinking like **Euler's method**, allow each tangent line to "re-aim" your movement after each small step. You should see two circles. Now verify your suspicions algebraically:

$$\frac{dy}{dx}=\frac{-x}{y} \Rightarrow y\,dy=-x\,dx \Rightarrow \int y\,dy=\int -x\,dx \Rightarrow \frac{y^2}{2}=\frac{-x^2}{2}+C \Rightarrow y^2+x^2=C$$

So the C in the algebra solution is simply the r^2 from our suspicion about the shape. **Notice that I did not claim the solution merely from the shapes suggested by the slope field, I used algebra to justify my answer while allowing the suspicion to guide my solution.**

BC note: In my opinion, explorations of slope fields and Euler's method are perfectly suited for a unit of study. Slope fields grant a rough picture which Euler's attempts to more carefully quantify.

What about initial conditions?

An initial condition simply identifies an ordered pair through which a particular solution to the differential equation passes. While a slope field visually represents ***all*** solutions to a differential equation, identifying an initial condition asks for the particular function (a specific "C") passing through that point.

Looking back at the slope field for the differential equation, $\frac{dy}{dx}=\frac{-x}{y}$, identify again the solution curve passing through **(1,0)**. If a question asked for the equation of the curve through **(1,0)** satisfying the differential equation, looking at the slope field and saying $x^2+y^2=1$ from the picture would not be sufficient. Graphs are suggestions and guides – **not proofs**. Rather, one would have to algebraically solve the differential equation and substitute:

$$\left.\frac{dy}{dx}=\frac{-x}{y}\right|_{(x,y)=(1,0)} \Rightarrow \left.x^2+y^2=C\right|_{(x,y)=(1,0)} \Rightarrow 1^2+0^2=C \Rightarrow x^2+y^2=1$$

You get the answer you suspected, but you used an algebraic proof guided by your informed intuitive knowledge of the slope field.

****Another potential use of slope fields in a multiple choice situation** would be the presentation of a slope field without its defining differential equation and asking which of the given functions could satisfy the unstated differential equation. Graphing the given functions would reveal whether they "followed" the pattern displayed in the slope field.

Are solutions unique?

Almost always! From a slope field point of view, knowing where to start tells me where to go. This does not mean that I will always be able to solve a differential equation algebraically, but I can always have a goc idea about what the solution looks like! It is important to note that since the solutions are generally unique ("always" is probably safe as far as the AP is concerned), then there can only be one solution curve passing through any given ordered pair in a slope field. **If two solution curves to a single differential equation intersect on its slope field, you probably have done something wrong**. Don't confuse tight approaches to an asymptote with intersections.

Remember – these are SLOPE fields

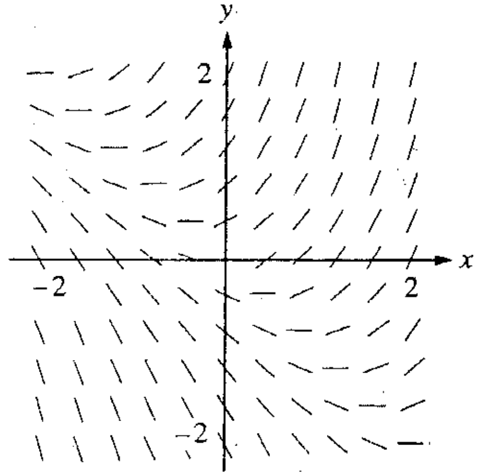
Since one is graphing *slopes*, the differential equation will always be first order. These are not "curvature" or "concavity" fields, although you can "see" the concavity in the slope field images. Again, **seeing concavity** is insufficient proof of the existence of that concavity. Algebraic means are required for proof. As multiple choice questions do not require work, students may be able to push their visual intuition further on these questions than on free response queries.

Released AP Calculus BC questions on slope fields

As of this year, slope fields are only a BC topic, although they will be added to the AB curriculum within two years. Also, since they are a relatively recent addition to BC (due to the 1990's calculus reform movement), there aren't that many questions. The only multiple choice example is 1998[BC24] and the free response questions are 1998[BC4], 2000[BC6], and 2002[BC5].

I'm also including the problem set from slope fields section of the Harvard Calculus text to give additional examples.

1998



24. Shown above is a slope field for which of the following differential equations?

(A) $\frac{dy}{dx} = 1 + x$ (B) $\frac{dy}{dx} = x^2$ (C) $\frac{dy}{dx} = x + y$ (D) $\frac{dy}{dx} = \frac{x}{y}$ (E) $\frac{dy}{dx} = \ln y$

1. The slope field for the equation $y' = x + y$ is shown in Figure 10.15.
   (a) Carefully sketch the solutions that pass through the points
   (i) $(0,0)$ (ii) $(-3,1)$ (iii) $(-1,0)$
   (b) From your sketch, write the equation of the solution passing through $(-1,0)$.
   (c) Verify your solution to part (b) by substituting it into the differential equation.

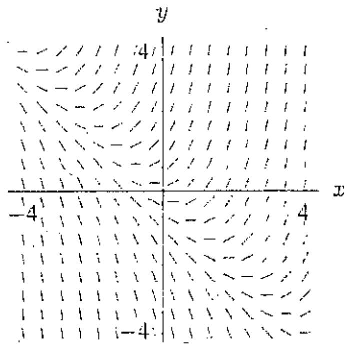


Figure 10.15: Slope field for $y' = x + y$

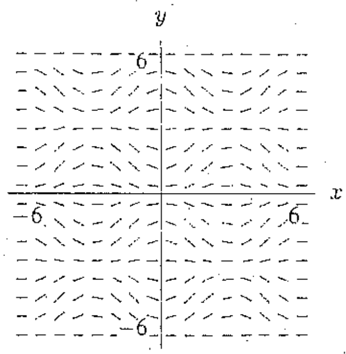


Figure 10.16: Slope field for $y' = (\sin x)(\sin y)$

2. Figure 10.16 shows the slope field for the equation $y' = (\sin x)(\sin y)$.
   (a) Carefully sketch the solutions that pass through the points: (i) $(0,-2)$ (ii) $(0,\pi)$.
   (b) What is the equation of the solution that passes through $(0, n\pi)$, where n is any integer?

3. Match the slope fields in Figure 10.17 with their differential equations:
   (a) $y' = -y$ (b) $y' = y$ (c) $y' = x$
   (d) $y' = 1/y$ (e) $y' = y^2$

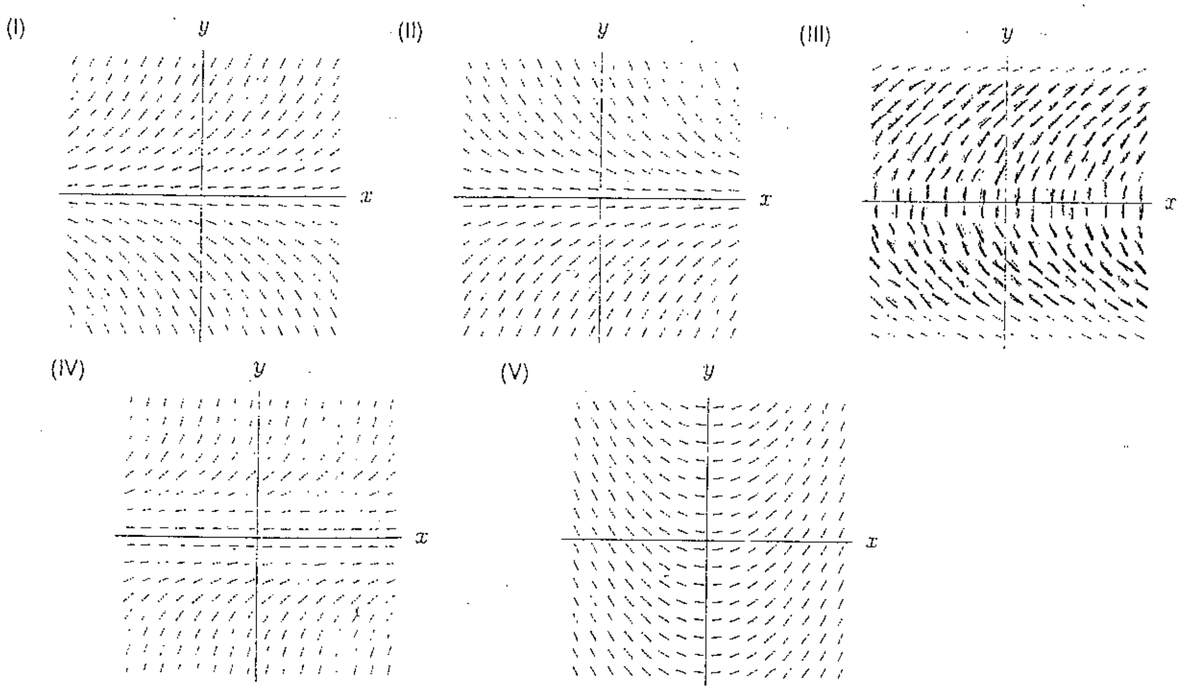


Figure 10.17

Hughes-Hallett, D., A. M. Gleason, et al. (1998). Calculus: Single and Multivariable. New York, Wiley & Sons, 498-501.

4. Match the slope fields in Figure 10.18 with their differential equations:

(a) $y' = 1 + y^2$ (b) $y' = x$ (c) $y' = \sin x$
(d) $y' = y$ (e) $y' = x - y$ (f) $y' = 4 - y$

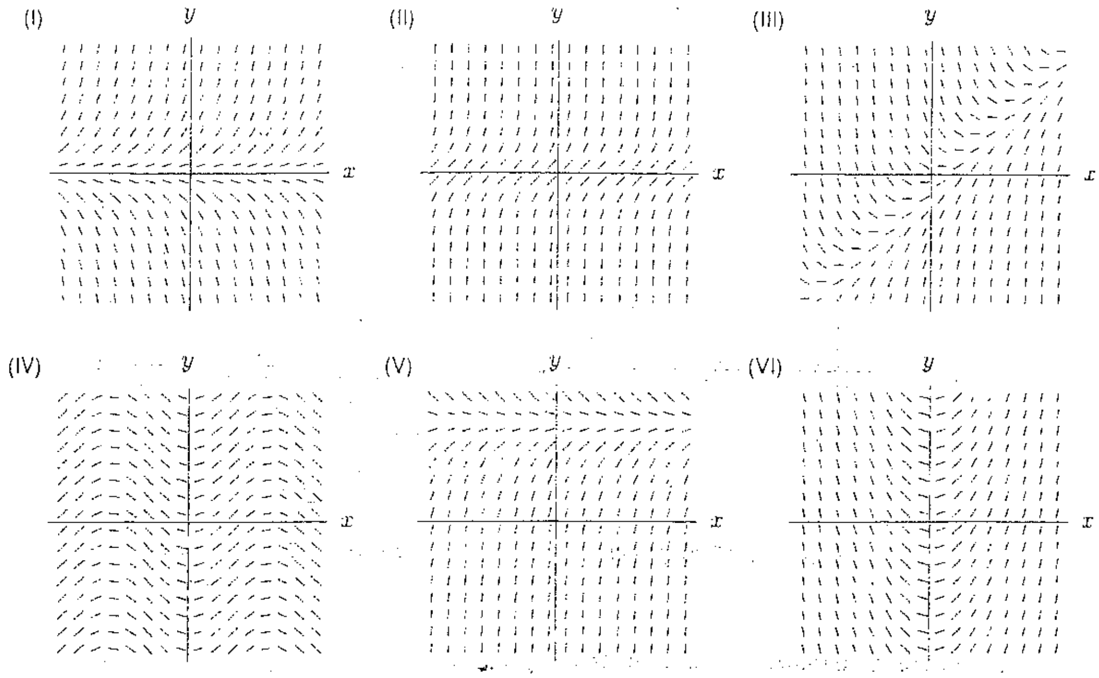


Figure 10.18: Each slope field is graphed for $-5 \le x \le 5$, $-5 \le y \le 5$

6. Match the slope fields in Figure 10.20 to the corresponding differential equations:

(a) $y' = xe^{-x}$ (b) $y' = \sin x$ (c) $y' = \cos x$
(d) $y' = x^2e^{-x}$ (e) $y' = e^{-x^2}$ (f) $y' = e^{-x}$

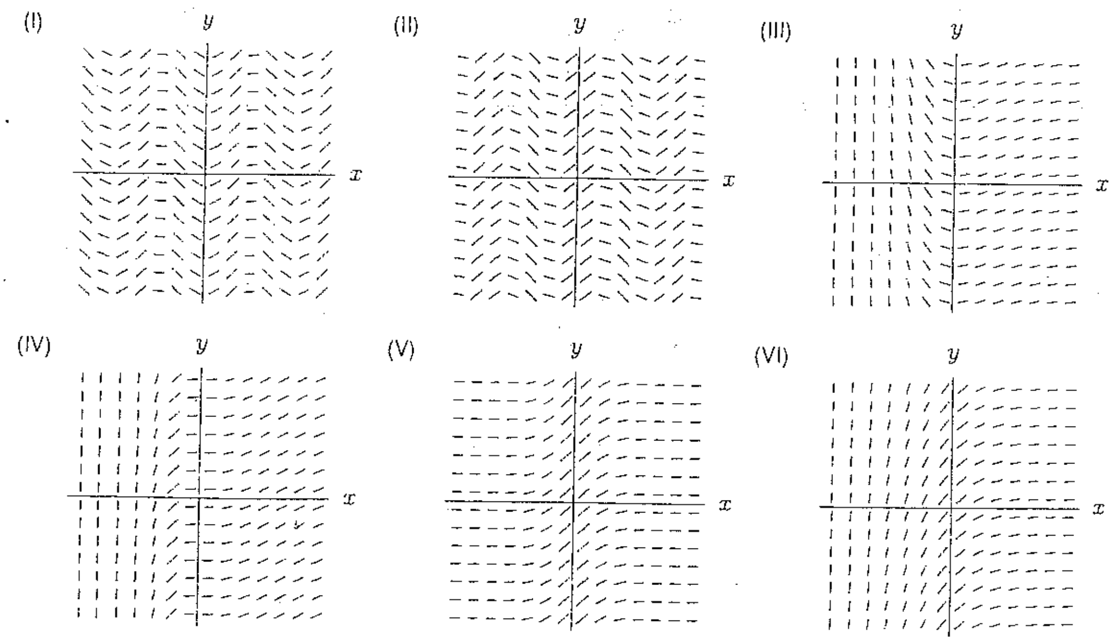


Figure 10.20

5. One of the slope fields in Figure 10.19 has the equation $y' = (x + y)/(x - y)$. Which one?

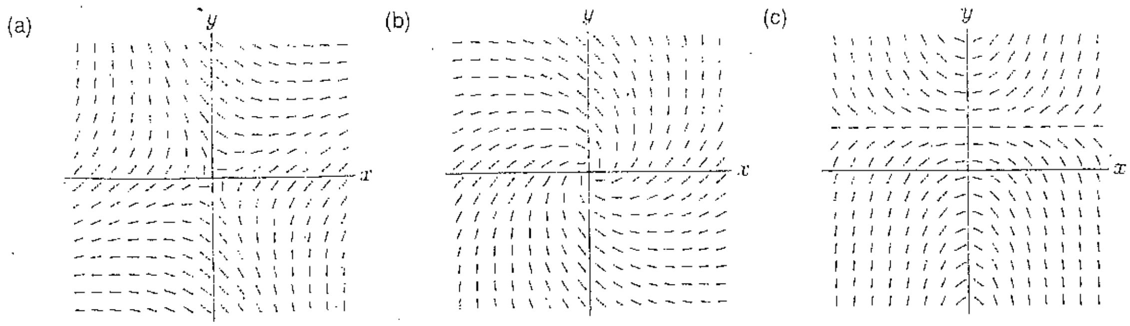


Figure 10.19

7. The slope field for $y' = 0.5(1 + y)(2 - y)$ is shown in Figure 10.21.

(a) Plot the following points on the slope field:

(i) the origin (ii) $(0, 1)$ (iii) $(1, 0)$
(iv) $(0, -1)$ (v) $(0, -5/2)$ (vi) $(0, 5/2)$

(b) Plot solution curves through the points in part (a).

(c) For which regions are all solution curves increasing? For which regions are all solution curves decreasing? When can the solution curves have horizontal tangents? Explain why, using both the slope field and the differential equation.

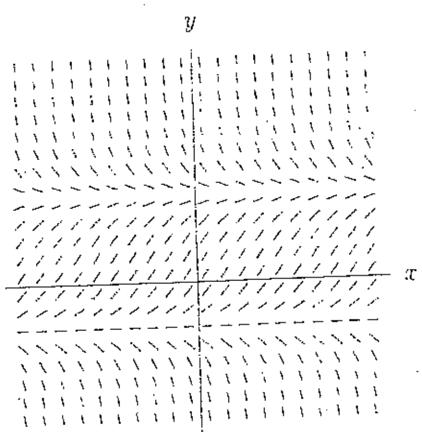


Figure 10.21: Note: x and y scales are equal

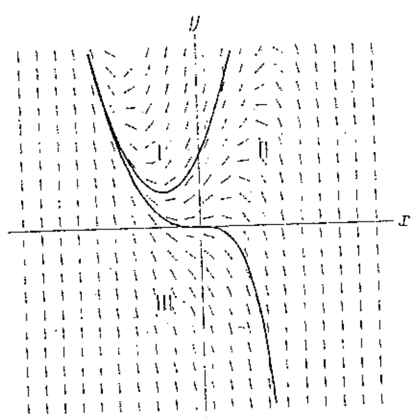


Figure 10.22

8. The slope field for the differential equation $y' = y - x^2$ together with two special solutions are shown in Figure 10.22.

(a) The two curves shown in the figure are $y = x^2 + 2x + 2$ and $y = x^2 + 2x + 2 - 2e^x$. Verify that these formulas give solutions, and indicate which formula goes with which curve.

(b) Sketch a few solutions to the differential equation in each of the three regions.

(c) Describe the solution curves you have sketched in region I.

(d) Describe the solution curves in region II.

(e) Describe the solution curves in region III.