Quick Review 9.3 *(For help, go to Sections 3.3 and 3.6.)*

In Exercises 1–5, find the smallest number M that bounds $|f|$ from above on the interval I (that is, find the smallest M such that $|f(x)| \le M$ for all x in I).

1. $f(x) = 2\cos(3x), \quad I = [-2\pi, 2\pi]$

2. $f(x) = x^2 + 3, \quad I = [1, 2]$

3. $f(x) = 2^x, \quad I = [-3, 0]$

4. $f(x) = \dfrac{x}{x^2+1}, \quad I = [-2, 2]$

5. $f(x) = \begin{cases} 2 - x^2, & x \le 1, \\ 2x - 1, & x > 1, \end{cases} \quad I = [-3, 3]$

In Exercises 6–10, tell whether the function has derivatives of all orders at the given value of a.

6. $\dfrac{x}{x+1}, \quad a = 0$

7. $|x^2 - 4|, \quad a = 2$

8. $\sin x + \cos x, \quad a = \pi$

9. $e^{-x}, \quad a = 0$

10. $x^{3/2}, \quad a = 0$

Section 9.3 Exercises

In Exercises 1–5, find the Taylor polynomial of order four for the function at $x = 0$, and use it to approximate the value of the function at $x = 0.2$.

1. e^{-2x}

2. $\cos(\pi x/2)$

3. $5\sin(-x)$

4. $\ln(1 + x^2)$

5. $(1 - x)^{-2}$

In Exercises 6–10, find the Maclaurin series for the function.

6. $\sin x - x + \dfrac{x^3}{3!}$

7. xe^x

8. $\cos^2 x \quad \left(= \dfrac{1 + \cos 2x}{2}\right)$

9. $\sin^2 x$

10. $\dfrac{x^2}{1 - 2x}$

11. Use graphs to find a Taylor polynomial $P_n(x)$ for $\ln(1 + x)$ so that $|P_n(x) - \ln(1 + x)| < 0.001$ for every x in $[-0.5, 0.5]$.

12. Use graphs to find a Taylor polynomial $P_n(x)$ for $\cos x$ so that $|P_n(x) - \cos x| < 0.001$ for every x in $[-\pi, \pi]$.

13. Find a formula for the truncation error if we use $P_6(x)$ to approximate $\dfrac{1}{1 - 2x}$ on $(-1/2, 1/2)$.

14. Find a formula for the truncation error if we use $P_9(x)$ to approximate $\dfrac{1}{1 - x}$ on $(-1, 1)$.

In Exercises 15–18, use the Lagrange form of the remainder to prove that the Maclaurin series converges to the generating function from the given exercise.

15. Exercise 7

16. Exercise 6

17. Exercise 9

18. Exercise 8

19. For approximately what values of x can you replace $\sin x$ by $x - (x^3/6)$ with an error magnitude no greater than 5×10^{-4}? Give reasons for your answer.

20. If $\cos x$ is replaced by $1 - (x^2/2)$ and $|x| < 0.5$, what estimate can be made of the error? Does $1 - (x^2/2)$ tend to be too large or too small? Support your answer graphically.

21. How close is the approximation $\sin x \approx x$ when $|x| < 10^{-3}$? For which of these values of x is $x < \sin x$? Support your answer graphically.

22. The approximation $\sqrt{1 + x} \approx 1 + (x/2)$ is used when x is small. Estimate the maximum error when $|x| < 0.01$.

23. The approximation $e^x \approx 1 + x + (x^2/2)$ is used when x is small. Use the Remainder Estimation Theorem to estimate the error when $|x| < 0.1$.

24. ***Hyperbolic sine and cosine*** The hyperbolic sine and hyperbolic cosine functions, denoted sinh and cosh respectively, are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

(Appendix A6 gives more information about hyperbolic functions.)

Find the Maclaurin series generated by $\sinh x$ and $\cosh x$.

25. ***(Continuation of Exercise 24)*** Use the Remainder Estimation Theorem to prove that $\cosh x$ equals its Maclaurin series for all real numbers x.

26. **Writing to Learn** Review the statement of the Mean Value Theorem (Section 4.2) and explain its relationship to Taylor's Theorem.

Quadratic Approximations Just as we call the Taylor polynomial of order 1 generated by f at $x = a$ the *linearization* of f at a, we call the Taylor polynomial of order 2 generated by f at $x = a$ the *quadratic approximation* of f at a.

In Exercises 27–31, find (a) the linearization and (b) the quadratic approximation of f at $x = 0$. Then (c) graph the function and its linear and quadratic approximations together around $x = 0$ and comment on how the graphs are related.

27. $f(x) = \ln(\cos x)$

28. $f(x) = e^{\sin x}$

29. $f(x) = 1/\sqrt{1 - x^2}$

30. $f(x) = \sec x$

31. $f(x) = \tan x$

32. Use the Taylor polynomial of order 2 to find the quadratic approximation of $f(x) = (1 + x)^k$ at $x = 0$ (k a constant). If $k = 3$, for approximately what values of x in the interval $[0, 1]$ will the magnitude of the error in the quadratic approximation be less than $1/100$?

33. ***A Cubic Approximation of e^x*** The approximation

$$e^x \approx 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

is used on small intervals about the origin. Estimate the magnitude of the approximation error for $|x| \le 0.1$.

34. ***A Cubic Approximation*** Use the Taylor polynomial of order 3 to find the *cubic approximation* of $f(x) = 1/(1 - x)$ at $x = 0$. Give an upper bound for the magnitude of the approximation error for $|x| \le 0.1$.

35. Consider the initial value problem,

$$\frac{dy}{dx} = e^{-x^2} \quad \text{and} \quad y = 2 \quad \text{when } x = 0.$$

(a) Can you find a formula for the function y that does not involve any integrals?

(b) Can you represent y by a power series?

(c) For what values of x does this power series actually equal the function y? Give a reason for your answer.

36. (a) Construct the Maclaurin series for $\ln(1 - x)$.

(b) Use this series and the series for $\ln(1 + x)$ to construct a Maclaurin series for

$$\ln\frac{1 + x}{1 - x}.$$

37. ***Identifying Graphs*** Which well-known functions are approximated on the interval $(-\pi/2, \pi/2)$ by the following Taylor polynomials?

(a) $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + \dfrac{62x^9}{2835}$

(b) $1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \dfrac{61x^6}{720} + \dfrac{277x^8}{8064}$

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

38. **True or False** The degree of the linearization of a function f at $x = a$ must be 1. Justify your answer.

39. **True or False** If $\sum_{n=0}^{\infty} \dfrac{x^{n+1}}{n!} = x + x^2 + \dfrac{x^3}{2!} + \cdots$ is the Maclaurin series for the function $f(x)$, then $f'(0) = 1$. Justify your answer.

40. **Multiple Choice** Which of the following gives the Taylor polynomial of order 5 approximation to $\sin(1.5)$?

(A) 0.965 (B) 0.985 (C) 0.997 (D) 1.001 (E) 1.005

41. **Multiple Choice** Let $\sum_{n=0}^{\infty} \dfrac{x^{n+1}}{n!} = x + x^2 + \dfrac{x^3}{2!} + \cdots$ be the Maclaurin series for $f(x)$. Which of the following is $f^{(12)}(0)$, the 12th derivative of f at $x = 0$?

(A) $1/11!$ (B) $1/12!$ (C) 0 (D) 1 (E) 12

42. **Multiple Choice** Let $\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!} = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots$

be the Maclaurin series for $\cos x$. Which of the following gives the smallest value of n for which $|P_n(x) - \cos x| < 0.01$ for all x in the interval $[-\pi, \pi]$?

(A) 12 (B) 10 (C) 8 (D) 6 (E) 4

43. **Multiple Choice** Which of the following is the quadratic approximation for $f(x) = e^{-x}$ at $x = 0$?

(A) $1 - x + \frac{1}{2}x^2$ (B) $1 - x - \frac{1}{2}x^2$

(C) $1 + x + \frac{1}{2}x^2$ (D) $1 + x$ (E) $1 - x$

Explorations

44. **Group Activity** Try to reinforce each other's ideas and verify your computations at each step.

(a) Use the identity

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

to obtain the Maclaurin series for $\sin^2 x$.

(b) Differentiate this series to obtain the Maclaurin series for $2 \sin x \cos x$.

(c) Verify that this is the series for $\sin 2x$.

45. ***Improving Approximations to π***

(a) Let P be an approximation of π accurate to n decimal places. Check with a calculator to see that $P + \sin P$ gives an approximation correct to $3n$ decimal places!

(b) Use the Remainder Estimation Theorem and the Maclaurin series for $\sin x$ to explain what is happening in part (a). (*Hint:* Let $P = \pi + x$, where x is the error of the estimate. Why should $(P + \sin P) - \pi$ be less than x^3?)

46. ***Euler's Identities*** Use Euler's formula to show that

(a) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$, and

(b) $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$.